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Boundary Conditions and the Simulation of Low Mach Number Flows

Thomas Hagstrom
Institute for Computational Mechanics in Propulsion
Lewis Research Center
Cleveland, Ohio

and The University of New Mexico
Albuquerque, New Mexico

and

Jens Lorenz
The University of New Mexico
Albuquerque, New Mexico

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Thomas Hagstrom
Institute for Computational Mechanics in Propulsion
NASA Lewis Research Center
Cleveland, OH
and
Department of Mathematics and Statistics
The University of New Mexico
Albuquerque, NM

Jens Lorenz
Department of Mathematics and Statistics
The University of New Mexico
Albuquerque, NM

Abstract

We consider the problem of accurately computing low Mach number flows, with the specific intent of studying the interaction of sound waves with incompressible flow structures, such as concentrations of vorticity. This is a multiple time (and/or space) scales problem, leading to various difficulties in the design of numerical methods. In this paper, we concentrate on one of these difficulties - the development of boundary conditions at artificial boundaries which allow sound waves and vortices to radiate to the far field. Nonlinear model equations are derived based on assumptions about the scaling of the variables. We then linearize these about a uniform flow and systematically derive exact boundary conditions using transform methods. Finally, we compute useful approximations to the exact conditions which are valid for small Mach number and small viscosity.

1 Introduction

The generation and interaction of sound waves with complex fluid flows is of great interest both from the point of view of fundamental fluid mechanics and applied mathematics and from the point of view of practical applications. Many important examples of these phenomena occur at low Mach number. A short list includes sound generation and propagation in water and the interaction of sound waves with laminar flames. Some advantages of studying low Mach number flows are the absence of shocks and the clear separation of 'incompressible' flow features, namely vortex dynamics, and the sound field. A disadvantage is the multiple time scales, requiring accurate computation for very long times as measured on the 'fast' scale. As a result, the numerical analysis of low Mach number flow is somewhat undeveloped in comparison with analytical theories.

Among the particular difficulties in the construction of accurate and efficient numerical methods are the time stepping scheme and the choice of radiation boundary conditions at artificial boundaries. The goal in choosing the time stepping procedure is to somehow exploit the simplicity of the equations governing the fast dynamics (essentially the wave equation) to allow large time steps. The problem with boundary conditions, which is the main topic of this work, is to find computationally usable procedures (for example local in time conditions) which are accurate over long times.

Essentially all problems in aero or hydroacoustics involve the radiation of energy to the far field in the form of sound waves and vortices. Any computational study then requires the introduction of artificial boundaries where accurate boundary conditions must be imposed. There is a vast literature concerning the construction of such conditions. (See, e.g. [4] and the references contained therein.) This literature is essentially divided into separate streams treating either hyperbolic problems or dissipative problems such as advection-diffusion equations. Our equations involve the coupling of a hyperbolic system governing the sound waves and an advection-diffusion equation for the vorticity, again acting on different time scales.

The remainder of the paper is organized as follows. In Section 2 we present the scalings we assume hold for the isentropic Navier-Stokes system and use them to derive a somewhat simplified set of nonlinear model equations. These are then linearized about a uniform flow for the purpose of systematically deriving exact and approximate (in the small Mach number - large Reynolds number limit) boundary conditions. The construction is carried out in Section 3. The result is a reasonably simple set of boundary conditions which combine 'standard' boundary operators of hyperbolic and advection-diffusion type.

2 Scalings and Model Equations

Consider the Navier-Stokes equations for conservation of mass and momentum in a fluid at constant entropy in two space dimensions:

$$\rho_t + u\rho_x + v\rho_y + \rho u_x + \rho v_y = 0, \tag{2.1}$$

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = \frac{\mu}{\rho}\left(\frac{4}{3}u_{xx} + u_{yy} + \frac{1}{3}v_{xy}\right), \tag{2.2}$$

$$v_t + uv_x + vv_y + \frac{1}{\rho}p_y = \frac{\mu}{\rho}\left(\frac{4}{3}v_{yy} + v_{xx} + \frac{1}{3}u_{xy}\right). \tag{2.3}$$

These must be supplemented by an equation of state relating pressure and density. For simplicity in the present work we use the equation of state for a γ-law gas:

$$p = K\rho^{\gamma}, \tag{2.4}$$

though eventually other equations of state, for example ones appropriate for liquids, will be included in the model.

To put the equations in nondimensional form we introduce a characteristic length scale, L, fluid velocity, S, and fluid pressure, P. From the latter we deduce a characteristic density, $D = (P/K)^{1/\gamma}$, and sound speed, $C^2 = \gamma P/D$. We then have two important dimensionless parameters, the Mach number which we take to be small and the Reynolds number which we take to be large:

$$M = \frac{S}{C} \ll 1, \quad Re = \frac{SLD}{\mu} \gg 1. \tag{2.5}$$

There are two natural time scales, based respectively on the fluid velocity and sound speed:

$$T_{\text{slow}} = \frac{L}{S}, \quad T_{\text{fast}} = \frac{L}{C} = M \cdot T_{\text{slow}}. \tag{2.6}$$

As we are interested in flows where 'fast' sound waves are present, we choose the latter as our characteristic time scale. We use the same letters for the dimenionless variables as for the dimensioned variables above, except for the fast time variable which we call τ to maintain notational consistency with our study of the one-dimensional problem [7]. (We reserve t for the slow time, $t = M\tau$.) We then have:

$$\rho_\tau + M\rho u_x + M\rho v_y + Mu\rho_x + Mv\rho_y = 0, \tag{2.7}$$

$$u_\tau + Muu_x + Mvu_y + \frac{1}{\gamma M\rho}p_x = \frac{M}{\rho Re}\left(\frac{4}{3}u_{xx} + u_{yy} + \frac{1}{3}v_{xy}\right), \tag{2.8}$$

$$v_\tau + Muv_x + Mvv_y + \frac{1}{\gamma M\rho}p_y = \frac{M}{\rho Re}\left(\frac{4}{3}v_{yy} + v_{xx} + \frac{1}{3}u_{xy}\right), \tag{2.9}$$

$$p = \rho^\gamma. \tag{2.10}$$

A glance at the momentum equations reveals potentially large $O(M^{-1})$ terms involving the pressure gradient. If indeed these terms were of that order we would expect the velocities to become large and the local Mach numbers to become $O(1)$. Therefore, if the Mach number is to be low throughout the flow, the pressure gradient must be $O(M)$. This leads us to introduce a new variable, q, which contains the pressure variations. That is:

$$p = 1 + \gamma Mq, \quad \rho = (1+\gamma Mq)^{1/\gamma} = 1 + Mq + O(M^2). \tag{2.11}$$

Substituting this into the dimensionless system and discarding terms $O(M^2)$ we finally obtain our nonlinear model system:

$$q_\tau + (1+Mq)u_x + (1+Mq)v_y + Muq_x + Mvq_y = 0, \tag{2.12}$$

$$u_\tau + (1-Mq)q_x + Muu_x + Mvu_y = \frac{M}{Re}\left(\frac{4}{3}u_{xx} + u_{yy} + \frac{1}{3}v_{xy}\right), \tag{2.13}$$

$$v_\tau + (1-Mq)q_y + Muv_x + Mvv_y = \frac{M}{Re}\left(\frac{4}{3}v_{yy} + v_{xx} + \frac{1}{3}u_{xy}\right). \tag{2.14}$$

Alternatively, we could have fixed the time scale, T, and then chosen from two natural spatial scales, CT and ST. This suggests the possibility of multiple spatial scales present in the solution, which is the case for many important aeroacoustic phenomena. To justify the approximations, we must assume that L is chosen so that derivatives are $O(1)$. For certain problems, for example the aeolian tones produced by flow past a cylinder, this implies that the sound waves will be slowly varying in space. We have not tried to make use of this in our derivation of boundary conditions.

It is of interest to compare this model system to the equations considered by other authors. Both Klainerman and Majda [9] and Kreiss, Lorenz and Naughton [11] have studied the incompressible limit, $M \to 0$. Then it is natural to take the slow time scale, where $\partial/\partial\tau$ is replaced by $M \cdot \partial/\partial t$, and to suppose that pressure variations scale like M^2. Then one obtains the incompressible Navier-Stokes equations by setting $M = 0$. This is a singular perturbation problem, as the incompressible equations require fewer initial conditions and fewer boundary conditions at inflow than the compressible equations. Hence, there is a possibility of boundary layers for small M, as analyzed in [11]. It is important that the conditions we develop do not generate such layers. Requirements on the initial data so that these 'nearly incompressible' scalings are maintained are studied both numerically and theoretically in [3].

Setting $M = 0$ in our case leads instead to a linear symmetric hyperbolic system governing the sound waves:

$$\begin{pmatrix} q \\ u \\ v \end{pmatrix}_\tau + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ u \\ v \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.15}$$

Our boundary conditions must also be accurate for this system, which is equivalent to the wave equation for q and the dilatation, $u_x + v_y$, coupled with the condition that vorticity be constant. (For finite M this means that vorticity is slowly varying.) In the next section we shall see that our boundary condition construction involves a standard (if unsatisfactorally solved) problem in the design of boundary conditions for the wave equation.

Finally, in order to carry out an analytic study of boundary conditions using transform methods, we linearize our model problem about a uniform flow, U, V. Again keeping the same letters for the linearized variables we have:

$$q_\tau + u_x + v_y + MUq_x + MVq_y = 0, \tag{2.16}$$

$$u_\tau + q_x + MUu_x + MVu_y = \frac{M}{Re}\left(\frac{4}{3}u_{xx} + u_{yy} + \frac{1}{3}v_{xy}\right), \tag{2.17}$$

$$v_\tau + q_y + MUv_x + MVv_y = \frac{M}{Re}\left(\frac{4}{3}v_{yy} + v_{xx} + \frac{1}{3}u_{xy}\right). \tag{2.18}$$

3 Boundary Conditions for the Model Problem

To explain our principle of deriving exact boundary conditions, we consider a general constant coefficient Cauchy problem of the form

$$w_t = P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})w, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0 \tag{3.1}$$

$$w(x, y, 0) = f(x, y). \tag{3.2}$$

It is assumed that $f \in L^2$ and that (3.1)-(3.2) is well-posed in L^2. (See [10] for a definition and discussion of this concept.) Also, we assume the initial data have compact support in x. More precisely, we assume that $f(x, y)$ is only nonzero for

$$-L + \delta \leq x \leq L - \delta \quad \text{where} \quad L > \delta > 0. \tag{3.3}$$

We want to replace the Cauchy problem by an initial-boundary value problem (IBVP) with boundary conditions at $x = \pm L$ so that the solution of the IBVP agrees with the solution of the Cauchy problem restricted to $-L \le x \le L$. Applying Fourier transformation in y and Laplace transformation in t to (3.1)-(3.2) we arrive at

$$s\hat{w}(x,k,s) - P(\frac{d}{dx}, ik)\hat{w}(x,k,s) = \hat{f}(x,k), \quad x \in \mathbb{R}. \tag{3.4}$$

For any fixed k, s, equation (3.4) is an ODE system in x, which is generally of mixed order. We want to derive boundary conditions at $x = \pm L$ which determine the L^2-solution of (3.4). To this end, we assume that (3.4) can be written as a first-order system

$$\frac{d}{dx}W - M(k,s)W = F(x,k), \quad x \in \mathbb{R}. \tag{3.5}$$

Well-posedness of (3.1)-(3.2) in L^2 implies that the symbol $P(ik_1, ik_2)$ of the partial differential operator P obeys an estimate

$$\|e^{P(ik_1,ik_2)t}\| \le Ke^{\alpha t}, \quad k_1, k_2 \in \mathbb{R}, \quad t \ge 0. \tag{3.6}$$

One can then prove that the system matrix $M = M(k,s)$ in (3.5) has no purely imaginary eigenvalues for $\mathrm{Re}\, s > \alpha$. Consequently, for $\mathrm{Re}\, s > \alpha$, the system can be transformed to a block form in which the exponentially growing modes (in x) are separated from the exponentially decaying ones. This is the main idea to obtain the exact boundary conditions. We assume that M has dimension $D \times D$ with D_- eigenvalues in the left half-plane and $D_+ = D - D_-$ eigenvalues in the right half-plane. We further assume, for simplicity, that M can be diagonalized. A nonzero row vector ψ of dimension D is called a left eigenvector of M if $\psi M = \lambda \psi$.

Theorem. Let $\mathrm{Re}\, s > \alpha$ and assume that $M(k,s)$ has a complete set of eigenvectors. Suppose that

$$\psi^j = \psi^j(k,s), \quad j = 1, \dots, D_-, \tag{3.7}$$

are linearly independent left eigenvectors of $M(k,s)$ corresponding to eigenvalues with negative real parts. Similarly, suppose that

$$\psi^j = \psi^j(k,s), \quad j = D_- + 1, \dots, D, \tag{3.8}$$

are linearly independent left eigenvectors of $M(k,s)$ corresponding to eigenvalues with positive real parts. The boundary conditions

$$\psi^j(k,s)W = 0 \quad \text{at } x = -L, \quad j = 1, \dots, D_-, \tag{3.9}$$

$$\psi^j(k,s)W = 0 \quad \text{at } x = L, \quad j = D_- + 1, \dots, D, \tag{3.10}$$

determine a unique solution of the system (3.5), namely the unique solution in $L^2(\mathbb{R})$. Inversion of the Fourier-Laplace transform leads to the solution of the Cauchy problem.

We now apply the general theory to the model system (2.16)-(2.18) where we assume $U > 0$. (Hence, $x = -L$ corresponds to an inflow boundary and $x = L$ to an outflow boundary of the underlying uniform flow.) Since the exact boundary conditions are independent of the initial data, the data are ignored in the following. Fourier-Laplace transformation leads to

$$\sigma\hat{q} + \hat{u}_x + ik\hat{v} + MU\hat{q}_x + ikMV\hat{q} = 0, \tag{3.11}$$

$$\sigma\hat{u} + \hat{q}_x + MU\hat{u}_x + ikMV\hat{u} = M\nu\Big(\frac{4}{3}\hat{u}_{xx} - k^2\hat{u} + \frac{ik}{3}\hat{v}_x\Big), \tag{3.12}$$

$$\sigma\hat{v} + ik\hat{q} + MU\hat{v}_x + ikMV\hat{v} = M\nu\Big(- \frac{4}{3}k^2\hat{v} + \hat{v}_{xx} + \frac{ik}{3}\hat{u}_x\Big). \tag{3.13}$$

Here $\nu = 1/Re$, and $\sigma = Ms$ is the dual to the stetched time variable, $\tau = t/M$. To determine the dispersion relation, we make the usual ansatz

$$\hat{q} = e^{\lambda x}\phi_1, \quad \hat{u} = e^{\lambda x}\phi_2, \quad \hat{v} = e^{\lambda x}\phi_3, \tag{3.14}$$

and obtain the condition

$$det\, A(\lambda) = 0, \tag{3.15}$$

the coefficients of the 3×3 matrix $A(\lambda)$ being at most quadratic in λ. The polynomial $det\, A(\lambda)$ has degree 5 and factors into the product of the quadratic

$$Q_2(\lambda) = \sigma + ikMV + M\nu k^2 + MU\lambda - M\nu\lambda^2, \tag{3.16}$$

and the cubic

$$Q_3(\lambda) = \lambda^2 - k^2 - \Big(\sigma + ikMV + \frac{4M\nu}{3} + MU\lambda - \frac{4M\nu}{3}\lambda^2\Big)\Big(\sigma + ikMV + MU\lambda\Big). \tag{3.17}$$

(This factorization is not accidental. In fact, if one writes (2.16)-(2.18) in terms of $\omega = v_x - u_y$, $\delta = u_x + v_y$, and q, then one obtains a second-order equation for ω which is decoupled from a system for (q, δ). The quadratic equation $Q_2(\lambda) = 0$ is the dispersion relation for ω whereas $Q_3(\lambda) = 0$ is the dispersion relation for the (q, δ) system.)

Henceforth we assume

$$0 < \nu \ll M \ll 1, \quad \sigma = O(1), \quad k = O(1). \tag{3.18}$$

Then the solutions of $Q_2(\lambda) = 0$ are

$$\lambda_{1,2} \approx \frac{U}{2\nu}\Big(1 \pm (1 + \frac{2\nu\tilde{\sigma}}{MU^2})\Big), \tag{3.19}$$

with

$$\tilde{\sigma} = \sigma + ikMV. \tag{3.20}$$

Setting $M = 0$ in Q_3, one obtains the roots

$$\lambda_{3,4} \approx \pm\sqrt{\sigma^2 + k^2}. \tag{3.21}$$

The third root of Q_3 is

$$\lambda_5 \approx -\frac{3}{4M^2\nu U} - \frac{\tilde{\sigma}}{MU} + \frac{3U}{4\nu}. \tag{3.22}$$

To determine the exact conditions, we need to rewrite (3.11)-(3.13) as a first-order system for

$$W = (\hat{q}, \hat{u}, \hat{u}_x, \hat{v}, \hat{v}_x)^T \tag{3.23}$$

of the form

$$W_x = M(k, s)W, \tag{3.24}$$

and then to determine the left eigenvectors of M. The eigenvalues of M are precisely the roots $\lambda_1, \ldots, \lambda_5$. They satisfy the sign conditions:

$$\operatorname{Re}\lambda_j > 0 \quad \text{for} \quad j = 1, 3 \tag{3.25}$$

$$\operatorname{Re}\lambda_j < 0 \quad \text{for} \quad j = 2, 4, 5 \tag{3.26}$$

To leading order (recall the assumption (3.18)) one obtains for the corresponding left eigenvectors of M:

$$\psi^1 \approx \Big(ik, 0, 0, \tilde{\sigma}, MU\Big), \tag{3.27}$$

$$\psi^2 \approx \Big(ik, ikMU, 0, \tilde{\sigma}, 0\Big), \tag{3.28}$$

$$\psi^3 \approx \Big(-\sqrt{\sigma^2 + k^2}, \tilde{\sigma}, 0, -ikMU, 0\Big), \tag{3.29}$$

$$\psi^4 \approx \Big(\sqrt{\sigma^2 + k^2}, \tilde{\sigma}, 0, -ikMU, 0\Big), \tag{3.30}$$

$$\psi^5 \approx \Big(\tilde{\sigma}, -MU\sigma, 1, ik, 0\Big). \tag{3.31}$$

The corresponding boundary conditions for $j = 1, 2, 5$ can be transformed easily to physical space. One obtains (to the order given):
At $x = L$ (from $j = 1$):

$$q_y + v_\tau + MVv_y + MUv_x = 0. \tag{3.32}$$

At $x = -L$ (from $j = 2$):

$$q_y + MUu_y + v_\tau + MVv_y = 0. \tag{3.33}$$

At $x = -L$ (from $j = 5$):

$$q_\tau + MVq_y - MUu_\tau + u_x + v_y = 0 \tag{3.34}$$

To translate the conditions for $j = 3, 4$ into physical space, locally in time, we must approximate the root $\sqrt{\sigma^2 + k^2}$ by a rational function in σ. This difficulty, which is not unexpected, is precisely the one encountered in developing boundary conditions for the wave equation. Most approximations proposed in the literature are designed to be accurate in the $(k/\sigma) \to 0$ limit. However, we also want accuracy over long times, suggesting the use of an approximation accurate as $\sigma = Ms \to 0$. Recently, a number of papers have appeared dealing with the long time behavior of approximate boundary conditons for the wave equation. Barry, Bielak, and MacCamy [1] introduce a useful notion of dissipativity, where the large σ approximation is modified to avoid exponential error growth. Engquist and Halpern [2] propose conditions which are exact in both the $\sigma \to \infty$ and $\sigma \to 0$ limits, and prove that solutions satisfying these rapidly converge to the correct steady state. However, it has also been shown [6] that, for certain exterior problems in two space dimensions, good long time accuracy is *impossible* to attain with standard boundary conditions.

A simple approximation, appearing already in [2], is

$$\sqrt{\sigma^2 + k^2} \approx \sigma + |k|. \tag{3.35}$$

If $(\mathcal{F}g)(k)$ denotes the Fourier transform of $g(y)$, we define

$$Hg = \mathcal{F}^{-1}(|k|(\mathcal{F}g)(k)). \tag{3.36}$$

Then the conditions for $j = 3, 4$ translate to:
At $x = L$ (from $j = 3, \sqrt{\sigma^2 + k^2} \approx \sigma + |k|$):

$$-q_\tau - Hq + u_\tau + MVu_y - MUv_y = 0. \tag{3.37}$$

At $x = -L$ (from $j = 4, \sqrt{\sigma^2 + k^2} \approx \sigma + |k|$):

$$q_\tau + Hq + u_\tau + MVu_y - MUv_y = 0. \tag{3.38}$$

Note that the conditions are still nonlocal in y since H is not a local operator. Often, this does not lead to difficulties. For example, if the given problem is periodic in y and one uses a discrete Fourier method in y-direction, then the operator H is easily discretized.

Instead of the approximation (3.35) one can also try

$$\sqrt{\sigma^2 + k^2} \approx \frac{\sigma(\sigma + a) + a|k|}{\sigma + a}, \quad a > 0, \tag{3.39}$$

which goes over into (3.35) for $a \to \infty$. The approximate eigenvectors are multiplied through by $\sigma + a$, and one obtains in physical space:
At $x = L$ (from $j = 3$, (3.39)):

$$-aHq + (\frac{\partial}{\partial \tau} + a)(-q_\tau + u_\tau + MVu_y - MUv_y) = 0. \tag{3.40}$$

At $x = -L$ (from $j = 4$, (3.39)):

$$aHq + (\frac{\partial}{\partial \tau} + a)(q_\tau + u_\tau + MVu_y - MUv_y) = 0. \tag{3.41}$$

Here the first two terms of the large σ expansion of the exact condition are matched as is the $\sigma \to 0$ limit. The parameter a could be chosen to optimize the approximation. Although we have, in these conditions, been careful to capture the leading order behavior as $\sigma \to \infty$, we note that terms of the order $\nu\sigma/M$ were earlier neglected.

So far we have not investigated the well-posedness of the resulting IBVPs, nor carried out numerical experiments. In one space dimension, however, we have implemented similar boundary conditions. We note that the troublesome symbol, $\sqrt{\sigma^2 + k^2}$, reduces to σ in one dimension, and so requires no approximation. Therefore, the computations only test the accuracy of the small M and small ν approximations of the exact conditions. Though a complete description of these experiments has appeared elsewhere [7], it is worthwhile to show a typical example. Figure 1 contains graphs of q and u for an initial pressure pulse. We see the pulse break up into left- and right-moving sound waves which propagate through the boundary with no visible distortion.

In these computations we have used a reasonably fine mesh, 2401 points, and a CFL number (based on the sound speed) of .2. The underlying velocity field is in the positive x direction, so the left boundary is an inflow boundary and the right an outflow boundary. The boundary conditions are then given by:

$$u + q = u_x - u_\tau = 0, \quad x = -1/2; \tag{3.42}$$

$$u - q = 0, \quad x = 1/2. \tag{3.43}$$

These conditions correspond to the specialization of the $j = 3, 4, 5$ conditions to the one dimensional case. Second order finite differences were used in both the interior and at the boundary. The additional numerical boundary condition at $x = L$ was second order extrapolation of the outgoing characteristic variable $u + q$. Full details of these computations, including a number of other cases, comparisons with conditions derived from energy arguments, as well as higher order (in M) approximations, can be found in [7].

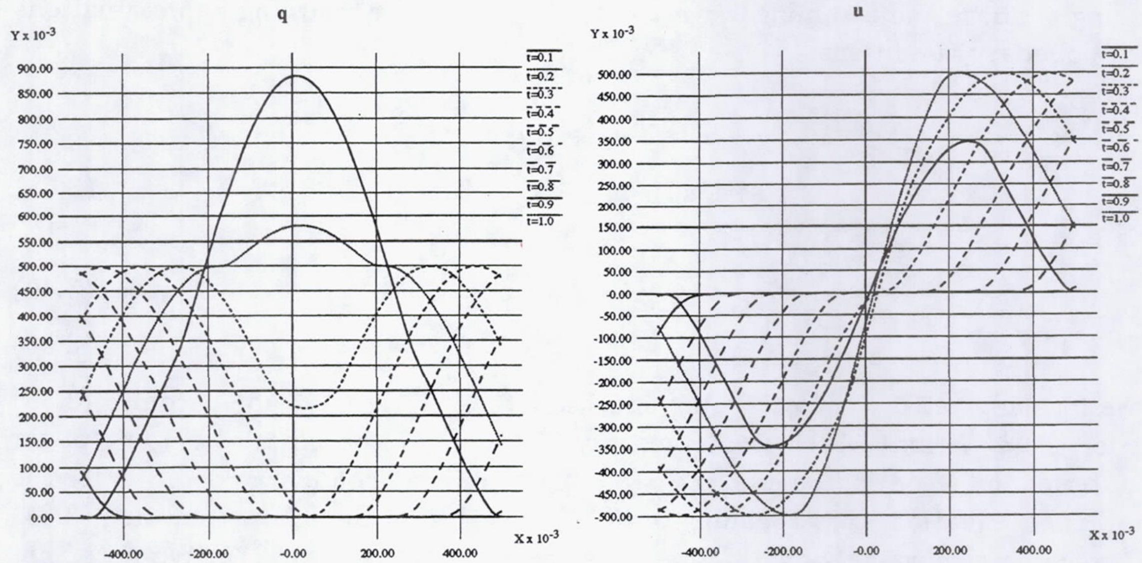

Figure 1: Plot of q and u, $M = .1$, $\nu = .01$.

It is interesting to compare these conditions to others which have appeared in the literature. Gustafsson and Stoor [5] proposed conditions based on energy arguments. Their purpose was to solve slightly compressible flow problems in the absence of sound waves, avoiding boundary layers at inflow. We do not expect their conditions to be accurate if sound waves are present, and our experiments in one space dimension confirm this. More in the spirit of this paper is Halpern's study of conditions for incompletely parabolic perturbations of hyperbolic systems [8]. The philosophy is essentially the same; derive expressions for exact conditions and approximate using the smallness of some parameters. The small parameters used in that study are ν and the tangential wave number. (In our case k.) This results in a somewhat different set of boundary relations. Our construction has the advantage of allowing $k = O(1)$, as the assumption $(k/\sigma) \ll 1$ may be difficult to justify. However, the conditions in [8] do not require $M \ll 1$.

It is also interesting to compare the derived boundary conditions for the slightly compressible model with boundary conditions derived by the same technique for the linearized incompressible equations:

$$u_t + U u_x + V u_y + p_x = \nu \Delta u, \tag{3.44}$$

$$v_t + Uv_x + Vv_y + p_y = \nu \Delta v, \tag{3.45}$$

$$u_x + v_y = 0. \tag{3.46}$$

Proceeding as above and assuming $0 < \nu << 1$, one obtains the following approximations to exact boundary conditions:

At $x = -L$:

$$Hp + u_t + Uu_x + Vu_y = 0, \tag{3.47}$$

$$p_y + Uu_y + v_t + Vv_y = 0. \tag{3.48}$$

At $x = L$:

$$-Hp + u_t + Uu_x + Vu_y = 0, \tag{3.49}$$

$$p_y + Uv_x + v_t + Vv_y = 0. \tag{3.50}$$

If we formally set $q = Mp, t = M\tau$ and use $u_x = -v_y$ then (3.45) becomes (3.38) except for the q_τ-term; (3.46) becomes exactly (3.33); (3.47) becomes (3.37) except for the $-q_\tau$-term; and (3.48) becomes exactly (3.32). There is no boundary condition for the incompressible equations corresponding to (3.34). Recall that the inflow condition (3.34) comes from the eigenvalue λ_5 with large negative real part (see (3.22)). We note that this correspondence requires that our approximation to $\sqrt{\sigma^2 + k^2}$ approach $|k|$ as $\sigma \to 0$.

To summarize, we've derived a nonlinear model system for the study of low Mach number flows with sound waves present and systematically derived approximate boundary conditions at inflow and outflow for linearizations of the model system about a uniform flow. The resulting equations display typical features of accurate conditions for the incompressible Navier-Stokes equations combined with a standard radiation condition for the wave equation. The latter, however, must be approximated so that long time accuracy is obtained. A complete study of the proposed conditions, including numerical experiments and analyses of well-posedness and the incompressible limit, are underway and will appear elsewhere.

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Thomas Hagstrom and Jens Lorenz

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Thomas Hagstrom, Institute for Computational Mechanics in Propulsion, NASA Lewis Research Center, and The University of New Mexico, Department of Mathematics and Statistics, Albuquerque, New Mexico 87131 (supported in part by NSF Grant DMS–9108072 and DOE Grant DE–FG03–92ER25128) and Jens Lorenz, The University of

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We consider the problem of accurately computing low Mach number flows, with the specific intent of studying the interaction of sound waves with incompressible flow structures, such as concentrations of vorticity. This is a multiple time (and/or space) scales problem, leading to various difficulties in the design of numerical methods. In this paper, we concentrate on one of these difficulties—the development of boundary conditions at artificial boundaries which allow sound waves and vortices to radiate to the far field. Nonlinear model equations are derived based on assumptions about the scaling of the variables. We then linearize these about a uniform flow and systematically derive exact boundary conditions using transform methods. Finally, we compute useful approximations to the exact conditions which are valid for small Mach number and small viscosity.

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